Signal Analysis & Communication ECE 355

Ch 5.3 – 5.7: Properties of DTFT (Contd.)

Ch 5.8: Linear Constant Coefficient Difference Equation. (LCCDE)

Lecture 27

16-11-2023

Ch. 5.3 – 5.7 PROPERTIES OF DTFT (contd.)

⑧ Differentiation in Frequency (useful in the example below) (partial fraction expansion)

$$-jn\,x[n] \longleftrightarrow \frac{d}{dw} X(e^{jw})$$

Proof

$$\text{RHS} \to \frac{d}{dw} \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} = \underbrace{\sum_{n=-\infty}^{\infty} x[n] (-jn)\, e^{-jwn}}_{\mathcal{F}\{x[n](-jn)\}}$$

⑨ Parseval's Relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} \underbrace{|X(e^{jw})|^2}_{\text{Energy Spectrum Density.}} dw$$

↳ Over a period as it is periodic in freq.-domain with period '2π'

⑩ Ch. 5.4. Convolution

$$x[n] * h[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})\, H(e^{jw})$$

Example

$$h[n] = a^n u[n], \quad |a| < 1, \quad a \neq 0$$

$$x[n] = b^n u[n], \quad |b| < 1, \quad b \neq 0$$

Find $y_{[n]}$.

$$y_{[n]} = x_{[n]} * h_{[n]}$$ Convolving would be cumbersome.

$$Y_{(e^{j\omega})} = \frac{1}{1-be^{-j\omega}} \cdot \frac{1}{1-ae^{-j\omega}}$$

- Use partial fraction expansion. (Appendix)

Cases:

I. If $a \neq b$

$$Y_{(e^{j\omega})} = \frac{A}{1-ae^{-j\omega}} + \frac{B}{1-be^{-j\omega}} \quad - (A)$$

$$A = (1-ae^{-j\omega}) Y_{(e^{j\omega})} \Big|_{e^{-j\omega}=\frac{1}{a}} = \frac{a}{a-b}$$

$$B = (1-be^{-j\omega}) Y_{(e^{j\omega})} \Big|_{e^{-j\omega}=\frac{1}{b}} = -\frac{b}{a-b}$$

So,

$$(A) \Rightarrow Y_{(e^{j\omega})} = \frac{a}{a-b} \times \frac{1}{1-ae^{-j\omega}} - \frac{b}{a-b} \times \frac{1}{1-be^{-j\omega}}$$

Applying inverse-DTFT

$$y_{[n]} = \frac{a}{a-b} a^n u_{[n]} - \frac{b}{a-b} b^n u_{[n]}$$

Using known DTFT pair.

$$a^n u_{[n]} \xleftrightarrow{F} \frac{1}{1-ae^{-j\omega}}$$
$(|a|<1)$

II. If $a=b$

$$Y(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})^2}$$

$$= \frac{j}{a}\, e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-ae^{-j\omega}}\right)$$

*

$$\updownarrow$$

$$-jna^n u[n]$$

$* \; x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$

time shift property

$n_0 = -1$ here

Applying inverse-DTFT,

$$y[n] = (n+1)\, a^n u[n+1]$$

NOTE:

- Although the RHS of $y[n]$ is multiplied by a step function that begins at $n=-1$, the sequence $(n+1)a^n u[n+1]$ is still zero prior to $n=0$, since $n+1=0$ at $n=-1$.
- Thus, $y[n]$ can alternately be expressed as:

$$y[n] = (n+1)\alpha^n u[n]$$

(11) Ch. 5.5. Multiplication

$$x_1[n]\, x_2[n] \longleftrightarrow X_1(e^{j\omega}) \underbrace{*_p}_{\text{Periodic Convolution.}} X_2(e^{j\omega})$$

$$\triangleq \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

(12) Ch. 5.7.2. Duality between DTFT & CTFS

Recall: - There is Duality b/w CTFT Synthesis & Analysis Eqns.
- Similarly, there is Duality b/w DTFS Synthesis & Analysis Eqns.
- However, there is no Duality b/w DTFT as well as as CTFS Synthesis & Analysis eqns.
- Instead, there exists Duality b/w DTFT & CTFS
- Discussed here!

- Recall Duality in CTFT

$$\boxed{\begin{array}{ll} \text{If} & x_{(t)} \overset{F}{\longleftrightarrow} g_{(\omega)} \\ \text{then} & g_{(t)} \longleftrightarrow 2\pi\, x_{(-\omega)} \end{array}}$$

- Now, look at the Analysis eqn. of DTFT

$$X_{(e^{j\omega})} = \sum_{n=-\infty}^{\infty} x_{[n]} e^{-j\omega n} \quad \text{— (B)}$$

Periodic with fundamental period '2π'

- Compare it with the synthesis eqn. of CTFS

$$f_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Now if this is periodic with period '2π' then $\omega_0 = \frac{2\pi}{2\pi} = 1$

- Therefore,

$$f_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jkt} \quad - (C)$$

- Comparing eqn (B) with eqn (C) and expressing eqn (B) as:

$$X_{(e^{j\omega})} = \sum_{n=-\infty}^{\infty} x_{[-n]} e^{jn\omega} \quad - (D)$$

Simply replacing 'n' with '-n'

- Comparing eqn (C) & eqn (D)

$$\underbrace{a_n = x_{[-n]}}_{\text{CTFS coefficients of } X_{(e^{j\omega})}} \quad , \quad k = n \in \mathbb{Z}$$

- Therefore,

$$X_{(e^{j\omega})} \xleftrightarrow{\mathcal{FS}} x_{[-n]}$$

- So finally we have.

$$x_{[n]} \xleftrightarrow{\mathcal{F}} X_{(e^{j\omega})} \xleftrightarrow{\mathcal{FS}} x_{[-n]}$$

NOTE : Check Table 5.3

Ch. 5.8. Linear Constant-Coefficient Difference Equation.

- A general linear constant-coefficient difference eqn. for an LTI sys. with I/P $x_{[n]}$ & O/P $y_{[n]}$ is of the form.

$$\sum_{k=0}^{N} a_k y_{[n-k]} = \sum_{k=0}^{M} b_k x_{[n-k]}$$

- Applying DTFT to both sides & using linearity & time shifting property.

$$\sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

$\gamma = e^{-j\omega}$

$$= \frac{\sum_{k=0}^{M} b_k \gamma^k}{\sum_{k=0}^{N} a_k \gamma^k}$$

Rational fraction in 'γ'

- Use "Partial Fraction Expansion" to obtain $h[n]$.
- Similar holds for finding $y[n]$ using convolution property ...